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Abstract: In congestion games, players use facilities from a common pool. The benefit that a player derives from using a facility depends, possibly among other things, on the number of users of this facility. The paper gives an easy alternative proof of the isomorphism between exact potential games and the set of congestion games introduced by Rosenthal (1973). It clarifies the relations between existing models on congestion games, and studies a class of congestion games where the sets of Nash equilibria, strong Nash equilibria and potential-maximizing strategies coincide. Particular emphasis is on the computation of potential-maximizing strategies.

Keywords: potential games, congestion, strong Nash equilibrium, potential-maximizing strategies

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1 Introduction

In recent years there has been a growing interest in the study of specific classes of non-cooperative games for which there exist pure strategy Nash equilibria. This paper deals with games derived from congestion models. In a congestion model, players use several facilities — also called machines or (primary) factors — from a common pool. The costs or benefits that a player derives from the use of a facility are, possibly among other factors, determined by the number of users of that same facility.

Congestion models can, for instance, be used to model the foraging behavior of a population of bees in a field of flowers. In deciding which flower to visit, each insect will take into account the quantity of nectar available and the number of bees already on the flower, because, as is intuitively clear, the more crowded the source of nectar, the less food is available per capita. In economics this kind of problems is studied in the literature on local public goods, where it is common to speak about "anonymous crowding" (cf. Wooders, 1989) to describe the negative externality arising from the presence of more than one user of the same facility. Another example is the problem faced by a set of unemployed workers who have to decide where to emigrate to get a job. The attraction of different countries depends on the conditions of the local labor market and, on the other hand, a crowding-out effect reduces the appeal of emigrating.

Rosenthal (1973) constitutes one of the pioneering papers on congestion games. In his model, each player chooses a subset of facilities. The benefit associated with each facility is a function only of the number of players using it. The payoff to a player is the sum of the benefits associated with each facility in his strategy choice, given the choices of the other players. Monderer and Shapley (1996) define exact potential games, games where information concerning the Nash equilibria can be incorporated in a potential function, a single real-valued function on the strategy space. Strategy profiles maximizing the potential are Nash equilibria of the potential game. By constructing a potential function for Rosenthal's congestion games, the existence of pure-strategy equilibria can be established.

Section 2 reviews some results on exact potential games. Section 3 describes Rosenthal's congestion games. Monderer and Shapley (1996) not only prove that every such congestion game is an exact potential game, but also establish that every exact potential game is isomorphic to a congestion game. Section 3 provides a new, considerably simpler proof of this result.

Konishi, Le Breton, and Weber (1997), Milchtaich (1996), and Quint and Shubik (1994) considered different classes of congestion games which in general do not admit a potential function, but were still able to prove the existence of pure Nash equilibria. Konishi, Le Breton, and Weber (1997), considering the same model as Milchtaich, have even shown the existence of a strong Nash equilibrium. Section 4 aims to clarify the relations and differences between these three classes of congestion games.

In Sections 5 and 6 we return to potential games and focus on a class of congestion games that combines features from the congestion models mentioned above. This class is shown to have interesting properties. Our main interest is in the relation between strong Nash equilibria and potential-maximizing strategies. In particular, it is shown that for each game in this class the set of strong Nash equilibria is nonempty and coincides with the set of Nash equilibria and the set of potential-maximizing strategies.

In Section 5 we analyze the geometric properties of this class of games, showing that it can be represented by a finitely generated cone. The aim of this section is twofold. First, it provides an easy way to compute potential-maximizing strategies, and second, it facilitates the proof that the sets of strong Nash equilibria, Nash equilibria, and potential-maximizing strategies are equal.

Implications of relaxing some of the assumptions underlying the congestion effect are discussed in Section 7.

Summarizing, this paper has four main goals:

- To provide a simple proof of the isomorphism between exact potential games and Rosenthal's congestion games in Section 3;
- To clarify the relations and differences between the models of Konishi *et al.* (1997), Milchtaich (1996), and Quint and Shubik (1994) in Section 4;
- To study the relation between potential-maximizing strategies and (strong) Nash

equilibria in a special class of congestion games, through the study of the structure of this class of games, which provides an easy way to find potential-maximizing strategies. This topic is taken up in Sections 5 and 6;

- To indicate the consequences of possible relaxation of several assumptions concerning the congestion effect in Section 7.

2 Exact potential games

This section defines exact potential games and surveys some results that are used in the remainder of the paper. A (strategic) game is a tuple $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$, where N is a nonempty, finite set of players, each player $i \in N$ has a nonempty, finite set X_i of pure strategies and a payoff function $u_i : \prod_{j \in N} X_j \to \mathbb{R}$ specifying for each strategy profile $x = (x_j)_{j \in N} \in \prod_{j \in N} X_j$ player i's payoff $u_i(x) \in \mathbb{R}$. Mixed strategies are not considered in this paper. Conventional game-theoretic notation is used: $X = \prod_{j \in N} X_j$ denotes the set of strategy profiles. Let $i \in N$. $X_{-i} = \prod_{j \in N \setminus \{i\}} X_j$ denotes the strategy profiles of i's opponents. Let $S \subseteq N$. $X_S = \prod_{j \in S} X_j$ denotes the set of strategy profiles of players in S. With a slight abuse of notation strategy profiles $x = (x_j)_{j \in N} \in X$ will be denoted by (x_i, x_{-i}) or $(x_S, x_{N \setminus S})$ if the strategy choice of player i or of the set S of players needs stressing.

Monderer and Shapley (1996) introduce exact potential games.

Definition 2.1 A strategic game $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ is an *exact potential game* if there exists a function $P : X \to \mathbb{R}$ such that for all $i \in N$, for all $x_{-i} \in X_{-i}$, and all $x_i, y_i \in X_i$:

$$u_i(x_i, x_{-i}) - u_i(y_i, x_{-i}) = P(x_i, x_{-i}) - P(y_i, x_{-i}).$$

The function P is called an *(exact) potential (function)* for G. $\triangleleft$

In other words, a strategic game is an exact potential game if there exists a real-valued function on the strategy space which exactly measures the difference in the payoff that accrues to a player if he unilaterally deviates.

	c	d
c	1,1	4,0
d	0,4	3,3

a

	L	R
T	0,2	2,3
B	2,5	4,6

b

Figure 1: Two exact potential games

Example 2.2 The Prisoner's Dilemma game of Figure 1a is an exact potential game with an exact potential function given by $P(c,c) = 5, P(c,d) = P(d,c) = 4, P(d,d) = 3$. The game in Figure 1b is an exact potential game with an exact potential function given by $P(T,L) = 0, P(T,R) = 1, P(B,L) = 2, P(B,R) = 3$. $\lhd$

If $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ has an exact potential P, the definition of an exact potential game immediately implies that the Nash equilibria of G and $\langle N, (X_i)_{i\in N}, (P)_{i\in N}\rangle$, the game obtained by replacing each payoff function by the potential P, coincide.

Proposition 2.3 *Let $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be an exact potential game. Then G has at least one (pure-strategy) Nash equilibrium.*

Proof. Let P be an exact potential for G. Since X is finite, $\arg\max_{x\in X} P(x)$ is a nonempty set. Clearly, all elements in this set are pure-strategy Nash equilibria. $\square$

Facchini *et al.* (1997) provide a characterization of exact potential games by splitting them up into coordination games and dummy games.

Definition 2.4 A game $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ is a

- *coordination game* if there exists a function $u : X \to \mathbb{R}$ such that $u_i = u$ for all $i \in N$;
- *dummy game* if for all $i \in N$ and all $x_{-i} \in X_{-i}$ there exists a $k \in \mathbb{R}$ such that $u_i(x_i, x_{-i}) = k$ for all $x_i \in X_i$.

$\lhd$

In a coordination game, players pursue the same goal, reflected by the identical payoff functions. In a dummy game, a player's payoff does not depend on his own strategy.

Theorem 2.5 *Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a strategic game. G is an exact potential game if and only if there exist functions $(c_i)_{i \in N}$ and $(d_i)_{i \in N}$ such that*

- *$u_i = c_i + d_i$ for all $i \in N$,*
- *$\langle N, (X_i)_{i \in N}, (c_i)_{i \in N} \rangle$ is a coordination game, and*
- *$\langle N, (X_i)_{i \in N}, (d_i)_{i \in N} \rangle$ is a dummy game.*

Proof. The 'if'-part is obvious: the payoff function of the coordination game is an exact potential function of G. To prove the 'only if'-part, let P be an exact potential for G. For all $i \in N$, $u_i = P + (u_i - P)$. Clearly, $\langle N, (X_i)_{i \in N}, (P)_{i \in N} \rangle$ is a coordination game. Let $i \in N, x_{-i} \in X_{-i}$, and $x_i, y_i \in X_i$. Then $u_i(x_i, x_{-i}) - u_i(y_i, x_{-i}) = P(x_i, x_{-i}) - P(y_i, x_{-i})$ implies $u_i(x_i, x_{-i}) - P(x_i, x_{-i}) = u_i(y_i, x_{-i}) - P(y_i, x_{-i})$. So $\langle N, (X_i)_{i \in N}, (u_i - P)_{i \in N} \rangle$ is a dummy game. $\square$

The difference between two exact potential functions of a game is a constant function.

Proposition 2.6 *Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a game with exact potential functions P and Q. Then $P - Q$ is a constant function.*

Proof. Let $i \in N$. By Theorem 2.5, $u_i - Q$ and $u_i - P$ do not depend on the strategy choice of player i. Hence $(P - Q) = (u_i - Q) - (u_i - P)$ does not depend on the strategy choice of player i. This holds for every player $i \in N$: $(P - Q)$ is a constant function. $\square$

Proposition 2.6 implies that the set of strategy profiles maximizing a potential function of an exact potential game does not depend on the particular potential function that is chosen. Potential-maximizing strategies were used in the proof of Proposition 2.3 to show that exact potential games have pure-strategy Nash equilibria. The potential maximizer, formally defined for an exact potential game $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ as

$$PM(G) = \{x \in X \mid x \in \arg\max_{y \in X} P(y) \text{ for some potential function } P \text{ of } G\}$$

can therefore,as suggested by Monderer and Shapley (1996), act as an equilibrium refinement tool. Peleg, Potters, and Tijs (1996) provide an axiomatic approach to potential-maximizing strategies.

3 Rosenthal's congestion model

In a congestion model, players use several facilities — also called machines or (primary) factors — from a common pool. The costs or benefits that a player derives from the use of a facility are, possibly among other factors, determined by the number of users of a facility. The purpose of this section is to describe the congestion model of Rosenthal (1973). In this model, each player chooses a subset of facilities. The benefit associated with each facility is a function only of the number of players using it. The payoff to a player is the sum of the benefits associated with each facility in his strategy choice, given the choices of the other players. By constructing an exact potential function for such congestion games, the existence of pure-strategy Nash equilibria can be established. Moreover, Monderer and Shapley (1996) showed that every exact potential game is isomorphic to a congestion game. Their proof is rather complex. In this section we present a different proof which is shorter and in our opinion more intuitive. In fact, we use the decomposition of exact potential games into dummy games and coordination games stated in Theorem 2.5 to decompose the problem into two subproblems. It is shown that each coordination game and each dummy game is isomorphic to a congestion game.

A *congestion model* is a tuple $\langle N, F, (X_i)_{i \in N}, (w_f)_{f \in F} \rangle$, where

- N is a nonempty, finite set of players;
- F is a nonempty, finite set of facilities;
- For each player $i \in N$, his collection of pure strategies X_i is a nonempty, finite family of subsets of F;
- For each facility $f \in F$, $w_f : \{1, \ldots, n\} \to \mathbb{R}$ is the benefit function of facility f, with $w_f(r), r \in \{1, \ldots, n\}$, the benefits to each of the users of facility f if there is a total of r users.

This gives rise to a *congestion game* $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ where N and $(X_i)_{i \in N}$ are as above and for $i \in N$, $u_i : X \to \mathbb{R}$ is defined thus: for each $x = (x_1, \ldots, x_n) \in X$, and each $f \in F$, let $n_f(x) = |\{i \in N : f \in x_i\}|$ be the number of users of facility f if the players choose x. Then $u_i(x) = \sum_{f \in x_i} w_f(n_f(x))$. This definition implies that each player derives benefit from the facilities he uses, with benefits depending only on the number of users of the facility. Notice that benefit functions can achieve negative values, representing costs of using a facility.

The main result from Rosenthal's paper, formulated in terms of exact potentials, is given in the next proposition. Its proof is straightforward and therefore omitted.

Proposition 3.1 *Let $\langle N, F, (X_i)_{i \in N}, (w_f)_{f \in F} \rangle$ be a congestion model and G its congestion game. Then G is an exact potential game. A potential function is given by $P : X \to \mathbb{R}$ defined for all $x = (x_i)_{i \in N} \in X$ as*

$$P(x) = \sum_{f \in \cup_{i \in N} x_i} \sum_{\ell=1}^{n_f(x)} w_f(\ell).$$

Since X is finite, the game has a Nash equilibrium in pure strategies.

Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ and $H = \langle N, (Y_i)_{i \in N}, (v_i)_{i \in N} \rangle$ be two strategic games with identical player set N. G and H are *isomorphic* if for all $i \in N$ there exists a bijection $\varphi_i : X_i \to Y_i$ such that

$$u_i(x_1, \ldots, x_n) = v_i(\varphi_1(x_1), \ldots, \varphi_n(x_n)) \text{ for all } (x_1, \ldots, x_n) \in X.$$

A congestion game where the facilities have non-zero benefits only if all players use it as part of their strategy choice is clearly a coordination game. Also, each coordination game can be expressed in this form, as shown in the proof of the next theorem.

Theorem 3.2 *Each coordination game is isomorphic to a congestion game.*

Proof. Let $G = \langle N, (X_i)_{i \in N}, (u)_{i \in N} \rangle$ be an n-player coordination game in which each player has payoff function u. Introduce for each $x \in X$ a different facility $f(x)$. Define the congestion model $\langle N, F, (Y_i)_{i \in N}, (w_f)_{f \in F} \rangle$ with $F = \cup_{x \in X} \{f(x)\}$, for each $i \in N$: $Y_i = \{g_i(x_i) \mid x_i \in X_i\}$ where $g_i(x_i) = \cup_{x_{-i} \in X_{-i}} \{f(x_i, x_{-i})\}$, and for each $f(x) \in F$:

$$w_{f(x)}(r) = \begin{cases} u(x) & \text{if } r = n \\ \\ 0 & \text{otherwise} \end{cases}$$

For each $x \in X$: $\cap_{i \in N} g_i(x_i) = \{f(x)\}$, so the game corresponding to this congestion model is isomorphic to G (where the isomorphisms map x_i to $g_i(x_i)$). □

The proof is illustrated with a simple example.

0,0	1,1
2,2	3,3

a

A	B
C	D

b

	$\{A, C\}$	$\{B, D\}$
$\{A, B\}$	0,0	1,1
$\{C, D\}$	2,2	3,3

c

Figure 2: A coordination game

Example 3.3 Consider the coordination game in Figure 2a. For each strategy profile we introduce a facility as in Figure 2b. These are the facilities that we want to be used by both players if they play the corresponding strategy profile. To do this, give each player in a certain row (column) all facilities mentioned in this row (column). For instance, the second strategy of the row player will correspond with choosing facility set $\{C, D\}$. Now indeed, if both players play their second strategy, facility D is used by both players and all other facilities have one or zero users. Defining the benefits of D in case of two simultaneous users to be 3 and in case of less users zero, we obtain the payoff $(3, 3)$ in the lower righthand corner of Figure 2c. Similar reasoning applies to the other cells. ◁

Consider a congestion game in which the benefits for a facility are non-zero only if it is used by a single player. If for each player, given the strategy choices of the other players, it holds that his benefits arise from using one and the same facility, irrespective of his own strategy choice, we have a dummy game. Also, as shown in the next theorem, each dummy game is isomorphic to a congestion game with this property.

Theorem 3.4 *Each dummy game is isomorphic to a congestion game.*

Proof. Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ be a dummy game. Introduce for each $i \in N$ and each $x_{-i} \in X_{-i}$ a different facility $f(x_{-i})$. Define the congestion model $\langle N, M, (Y_i)_{i \in N}, (w_f)_{j \in F} \rangle$

with $F = \cup_{i \in N} \cup_{x_{-i} \in X_{-i}} \{f(x_{-i})\}$, for each $i \in N$: $Y_i = \{h_i(x_i) \mid x_i \in X_i\}$ where

$$\begin{aligned} h_i(x_i) &= \{f(x_{-i}) \mid x_{-i} \in X_{-i}\} \\ &\quad \cup \{f(y_{-j}) \mid j \in N \setminus \{i\} \text{ and } y_{-j} \in X_{-j} \text{ is such that } y_i \neq x_i\}, \end{aligned}$$

and for each $f(x_{-i}) \in M$:

$$w_{f(x_{-i})}(r) = \begin{cases} u_i(x_i, x_{-i}) & \text{if } r = 1 \text{ (with } x_i \in X_i \text{ arbitrary)} \\ 0 & \text{otherwise} \end{cases}$$

For each $i \in N$, $\overline{x}_{-i} \in X_{-i}$, and $\overline{x}_i \in X_i$: i is the unique user of $f(\overline{x}_{-i})$ in $(h_j(\overline{x}_j))_{j \in N}$ and all other facilities in $h_i(\overline{x}_i)$ have more than one user. Why? Let $i \in N$, $\overline{x}_{-i} \in X_{-i}$, and $\overline{x}_i \in X_i$. Then $f(\overline{x}_{-i}) \in h_i(\overline{x}_i)$ and for each $j \in N \setminus \{i\}$: $f(\overline{x}_{-i}) \notin h_j(\overline{x}_j)$, so i is indeed the unique user of $f(\overline{x}_{-i})$ in $(h_j(\overline{x}_j))_{j \in N}$. Let $f \in h_i(\overline{x}_i), f \neq f(\overline{x}_{-i})$.

- If $f = f(y_{-i})$ for some $y_{-i} \in X_{-i}$, then $y_{-i} \neq \overline{x}_{-i}$ implies that $y_j \neq \overline{x}_j$ for some $j \in N \setminus \{i\}$, so $f = f(y_{-i}) \in h_j(\overline{x}_j)$.
- If $f = f(y_{-j})$ for some $j \in N \setminus \{i\}$ and $y_{-j} \in X_{-j}$ with $y_i \neq \overline{x}_i$, then $f = f(y_{-j}) \in h_j(\overline{x}_j)$.

In both cases f has more than one user. So the game corresponding to this congestion model is isomorphic to G (where the isomorphisms map x_i to $h_i(x_i)$). □

Once again, this argumentation is illustrated by an example.

0,2	1,2
0,3	1,3

a

α, γ	β, γ
α, δ	β, δ

b

	$\{\beta, \gamma, \delta\}$	$\{\alpha, \gamma, \delta\}$
$\{\alpha, \beta, \delta\}$	0,2	1,2
$\{\alpha, \beta, \gamma\}$	0,3	1,3

c

Figure 3: A dummy game

Example 3.5 Consider the dummy game in Figure 3a. Introduce a different facility for each profile of opponent strategies as in Figure 3b. Include a facility $f(x_{-i})$ in each strategy of each player, except for the strategies of players $j \in N \setminus \{i\}$ specified by the profile x_{-i}. For instance, facility α was introduced for the first column of player 2; then α is part of every strategy, except for the first column of player 2. This yields the strategies as in Figure 3c. Define benefits for multiple users equal to zero. No matter what player 1 does, if his opponent chooses his second strategy, the benefits to player 1 can be attributed to facility β. Assign benefit 1 to a single user of this facility. Similar reasoning for the other payoffs yields the isomorphic congestion game in Figure 3c. ◁

In the previous two theorems it was shown that coordination and dummy games are isomorphic to congestion games. Using the decomposition of Theorem 2.5 we obtain that every exact potential game is isomorphic to a congestion game.

Theorem 3.6 *Every exact potential game is isomorphic to a congestion game.*

Proof. Let $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ be an exact potential game. Split it into a coordination game and a dummy game as in Theorem 2.5 and take their isomorphic congestion games as in Theorems 3.2 and 3.4. Without loss of generality, take their facility sets disjoint. Construct a congestion game isomorphic to G by taking the union of the two facility sets, benefit functions as in Theorems 3.2 and 3.4, and strategy sets $Y_i = \{g_i(x_i) \cup h_i(x_i) \mid x_i \in X_i\}$. □

0,2	2,3
2,5	4,6

a

	$\{A,C\}\cup\{\beta,\gamma,\delta\}$	$\{B,D\}\cup\{\alpha,\gamma,\delta\}$
$\{A,B\}\cup\{\alpha,\beta,\delta\}$	0+0,0+2	1+1,1+2
$\{C,D\}\cup\{\alpha,\beta,\gamma\}$	2+0,2+3	3+1,3+3

b

Figure 4: Exact potential game and isomorphic congestion game

Example 3.7 The exact potential game in Figure 1b is the sum of the coordination game from Example 3.3 and the dummy game from Example 3.5. Combining the two isomorphic congestion games from these examples yields a congestion game isomorphic to the exact potential game. See Figure 4. ◁

4 Congestion games

The games introduced by Konishi, Le Breton, and Weber (1997), Milchtaich (1996), and Quint and Shubik (1994) are similar, in the sense that the utility functions of the players are characterized by a congestion effect. The various classes of games we discuss are identified by means of different sets of properties concerning the structure of the strategic interaction. In particular, Konishi *et al.* (1997) impose the following assumptions (P1)–(P4) on a game $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$.

(P1) There exists a finite set F such that $X_i = F$ for all players $i \in N$.

The set F is called the "facility set" and a strategy for player i is choosing an element of F.

(P2) For each strategy profile $x \in X$ and all players $i, j \in N$: if $x_i \neq x_j$ and $x'_j \in X_j$ is such that $x_i \neq x'_j$, then $u_i(x_j, x_{-j}) = u_i(x'_j, x_{-j})$.

Konishi *et al.* (1997) call this assumption *independence of irrelevant choices*: for each player $i \in N$ and each strategy profile x the utility of i will not be altered if the set of players that choose the same facility as player i is not modified.

Let $x \in X, f \in F$. Denote as before by $n_f(x)$ the number of users of facility f in the strategy profile x. Then the third property can be stated as follows:

(P3) For each player $i \in N$ and all strategy profiles $x, y \in X$ with $x_i = y_i$: if $n_f(x) = n_f(y)$ for all $f \in F$, then $u_i(x) = u_i(y)$.

This *anonymity* condition reflects the idea that the payoff of player i depends on the number of players choosing the facilities, rather than on their identity. The fourth assumption, called *partial rivalry*, states that each player i would not regret that other players, choosing the same facility, would select another one. Formally:

(P4) For each player $i \in N$, each strategy profile $x \in X$, each player $j \neq i$ such that $x_j = x_i$ and each $x'_j \neq x_i$: $u_i(x_j, x_{-j}) \leqq u_i(x'_j, x_{-j})$.

Although Milchtaich (1996) introduces his model in a slightly different way, the resulting class of games is the same. More specifically Milchtaich (1996) introduces the conditions (P1), (P4), and the following assumption:

(P2') For each player $i \in N$ and all strategy profiles x, y with $x_i = y_i = f$: if $n_f(x) = n_f(y)$, then $u_i(x) = u_i(y)$.

In other words the utility of player i depends only on the number of users of the facility that i has chosen. Assuming (P1), it is straightforward to prove that (P2') implies both (P2) and (P3). The converse implication is also true.

Lemma 4.1 *Any game $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ satisfying (P1), (P2), and (P3) satisfies (P2').*

Proof. Let $G = \langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ satisfy (P1), (P2), and (P3). Let $i \in N$, $x, y \in X$ such that $x_i = y_i = f$ and assume that $n_f(x) = n_f(y)$. If $|F| = 1$, (P2') follows directly. Otherwise, from repeated use of (P2), we know that for a fixed $g \neq x_i$, $u_i(x_i, x_{-i}) = u_i(x_i, x'_{-i})$ where for each $j \in N \setminus \{i\}$:

$$x'_j = \begin{cases} x_i & \text{if } x_j = x_i, \\ & \\ g & \text{otherwise,} \end{cases}$$

and that $u_i(x_i, y_{-i}) = u_i(x_i, y'_{-i})$, where for each $j \in N \setminus \{i\}$:

$$y'_j = \begin{cases} x_i & \text{if } y_j = x_i, \\ & \\ g & \text{otherwise.} \end{cases}$$

Notice that for each $h \in F$, $n_h(x_i, x'_{-i}) = n_h(x_i, y'_{-i})$. So (P3) implies $u_i(x_i, x'_{-i}) = u_i(x_i, y'_{-i})$. Therefore, $u_i(x_i, x_{-i}) = u_i(x_i, x'_{-i}) = u_i(x_i, y'_{-i}) = u_i(y_i, y_{-i})$. □

Konishi *et al.* (1997) and Milchtaich (1996) independently proved the following

Theorem 4.2 *Each game $\langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$ satisfying (P1), (P2), (P3) and (P4), possesses a pure-strategy Nash equilibrium.*

Recall that, given a game $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$, a strategy profile x is called a strong Nash equilibrium if for every $S \subseteq N$ and all strategy profiles $y_S \in X_S$, there is at least one player $i \in S$ such that $u_i(y_S, x_{-S}) \leqq u_i(x)$. The set of strong Nash equilibria of a game G is denoted by $SNE(G)$. In general, the existence of a strong Nash equilibrium is not guaranteed, but Konishi *et al.* (1997) show

Theorem 4.3 *For each game satisfying (P1), (P2) (P3) and (P4), the set of strong Nash equilibria is nonempty.*

Finally, we mention the model introduced by Quint and Shubik (1994), where the assumption that all players have the same set of facilities (as stated by (P1)) is relaxed.

(P1') There exists a finite set F such that $X_i \subseteq F$ for all players $i \in N$.

Assuming that (P1') holds, it is still easy to see that (P2') implies (P2) and (P3). But the analogon of Lemma 4.1 does not hold.

Example 4.4 Take $N = \{1, 2, 3\}$, $F = \{a, b, c\}$ and strategy sets $X_1 = \{a, b\}$, $X_2 = \{a\}$, $X_3 = \{a, c\}$. This game satisfies (P1'). Assumption (P3) imposes no additional requirements and (P2) requires that $u_1(b, a, a) = u_1(b, a, c)$ and $u_3(a, a, c) = u_3(b, a, c)$. This does not imply $u_2(a, a, c) = u_2(b, a, a)$, which is required by (P2'). $\triangleleft$

Quint and Shubik (1994) show

Theorem 4.5 *All strategic games satisfying (P1'), (P2') and (P4) possess a pure Nash equilibrium.*

Games in the classes considered so far not necessarily admit a potential function. Consider now the following *cross-symmetry* condition, which states that the payoffs on a certain facility are player-independent, provided that the number of users is the same.

(P5) For all strategy profiles $x, y \in X$ and all players $i, j \in N$: if $x_i = y_j = f$ and $n_f(x) = n_f(y)$, then $u_i(x) = u_j(y)$.

Notice that (P5) together with (P1) implies (P2'), and thus (P2) and (P3). Moreover, (P1) and (P5) guarantee the existence of a potential.

Theorem 4.6 *Each game satisfying (P1) and (P5) is an exact potential game.*

Proof. Let $G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle$ satisfy (P1) and (P5). For any $f \in F$ and $x, y \in X$ such that $n_f(x) = n_f(y)$, we have by (P5): if there are $i, j \in N$ such that $x_i = y_j = f$, then $u_i(x) = u_j(y)$. This shows that for all $f \in F$ there exists a benefit function $w_f : \{1, \ldots, n\} \to \mathbb{R}$ such that for all $x \in X$, if $x_i = f$, then $u_i(x) = w_f(n_f(x))$. This makes the game G a congestion game as defined in Section 3[2]. The result now follows from Proposition 3.1. □

Remark 4.7 The theorem still holds if (P1') is substituted for (P1). It also follows from Proposition 3.1, that the benefit functions $(w_f)_{f\in F}$ give rise to a potential $P : x \mapsto \sum_{f\in\cup_{i\in N}\{x_i\}} \sum_{\ell=1}^{n_f(x)} w_f(\ell)$. ◁

As can be seen in the Prisoner's Dilemma in Figure 1a, exact potential games do not in general possess a strong Nash equilibrium. The remainder of this paper focuses on games that admit an exact potential and have strong Nash equilibria. Therefore, attention is restricted to the class $\mathcal{C}$ of congestion games satisfying not only (P1) and (P5), but also (P4). So

$$\mathcal{C} = \{G = \langle N, (X_i)_{i\in N}, (u_i)_{i\in N}\rangle \mid G \text{ satisfies (P1), (P4), and (P5)}\}. \tag{1}$$

5 On the structure of the class $\mathcal{C}$

We analyze the structure of the class $\mathcal{C}$ defined in (1). Let $n \in \mathbb{N}$ be a number of players and F a finite set of facilities. Denote by $\mathcal{C}(F, n)$ the set of all games in $\mathcal{C}$ with n players and facility set F. Identifying each game $G \in \mathcal{C}(F, n)$ with a set of vectors $(w_f)_{f\in F}$ like in the proof of Theorem 4.6, it is shown that $\mathcal{C}(F, n)$ is a finitely generated cone in $(\mathbb{R}^n_+)^F$. The vector notation of the games simplifies the proofs of the theorems on strong equilibria and the potential maximizer presented in Sections 6 and 7.

Let $G \in \mathcal{C}(F, n)$. Recall from Theorem 4.6 that for every $f \in F$ there exists a function $w_f : \{1, \ldots, n\} \to \mathbb{R}$ such that for all $x \in X$, if $x_i = f$, then $u_i(x) = w_f(n_f(x))$. From

[2] Where we identify choosing a facility $f \in F$ with choosing facility set $\{f\} \subseteq F$.

(P4), we have for each $f \in F$ and $t \in \{1, \ldots, n-1\}$ that $w_f(t) \geqq w_f(t+1)$. For convenience and without loss of generality we assume that $w_f(t) \geqq 0$ for all $f \in F, t \in \{1, \ldots, n\}$. This means that the game $G \in \mathcal{C}(F, n)$ is described by $|F|$ vectors of the form $(w_f(1), \ldots, w_f(n)), f \in F$, each in the set $V = \{v = (v_1, \ldots, v_n) \in \mathbb{R}^n_+ \mid v_t \geqq v_{t+1}$ for all $t \in \{1, \ldots, n-1\}\}$.

Proposition 5.1 *The set V is a finitely generated cone in $\mathbb{R}^n_+$. The extreme directions of V are the vectors $\mathbf{b}^1, \mathbf{b}^2, \ldots, \mathbf{b}^n$ with $\mathbf{b}^i = (\underbrace{1, 1, 1, 1}_{i \text{ times}}, 0, \ldots, 0)$. Furthermore, $\dim(V) = n$.*

Proof. The vectors $\mathbf{b}^1 = (1, 0, 0, \ldots, 0)$, $\mathbf{b}^i = (\underbrace{1, 1, 1, 1}_{i \text{ times}}, 0, \ldots, 0)$,..., $\mathbf{b}^n = (1, 1, 1, 1, \ldots, 1)$ are elements of V and each vector $\mathbf{v} \in V$ can be uniquely written as a nonnegative combination of $\mathbf{b}^1, \mathbf{b}^2, \ldots, \mathbf{b}^n$. To show this, let $\mathbf{v} \in V$ and define

$$\mathbf{B}_n = \begin{bmatrix} \mathbf{b}^1 \\ \mathbf{b}^2 \\ \vdots \\ \mathbf{b}^n \end{bmatrix}$$

So $\mathbf{B}_n$ is the $n \times n$ matrix whose i-th row is $\mathbf{b}^i$. Since $\det(\mathbf{B}_n) = 1$, the equation $\alpha \mathbf{B}_n = \mathbf{v}$ has exactly one solution. Clearly, α is nonnegative because of the decreasingness property of $\mathbf{v}$. The set V is therefore the cone $C(\mathbf{B}_n)$ where $C(\mathbf{B}_n) := \{\alpha \mathbf{B}_n \mid \alpha \in \mathbb{R}^n_+\}$.

The extreme directions of the cone $C(\mathbf{B}_n)$ are the vectors $\mathbf{b}^i$, $i \in \{1, \ldots, n\}$. This cone has furthermore the property that its dimension is the number of extreme directions. In other words we have that $\dim\ C(\mathbf{B}_n) = rank(\mathbf{B}_n) = n$. □

Essentially we proved

Corollary 5.2 *The class of games $\mathcal{C}(F, n)$ can be identified with a cone in $(\mathbb{R}^n_+)^F$ and $\dim(\mathcal{C}(F, n)) = |F| \times n$.*

In the next example we consider an extreme game of $\mathcal{C}(F, n)$, i.e., a game with facility set F such that w_f is an extreme direction in the cone V for each $f \in F$.

Example 5.3 Let G be a game in $\mathcal{C}(\{f,g\},4)$ such that $w_f = (1,0,0,0)$ and $w_g = (1,1,0,0)$. Nash equilibria are either those strategy profiles in which one of the players chooses f and the other three g, or those in which both facilities are chosen by two players. These situations will be depicted

$$(\boxed{1},0,0,0)$$
$$(1,1,\boxed{0},0)$$

for the first case and

$$(1,\boxed{0},0,0)$$
$$(1,\boxed{1},0,0)$$

for the second one, where the numbers in the square boxes indicate the payoff received by each player choosing this facility. Notice furthermore that the players are interchangeable as suggested by the cross-symmetry condition (P5). One easily checks that all Nash equilibria are strong. ◁

6 Strong Nash equilibria and the potential maximizer

In this section it is shown that on the class $\mathcal{C}$, the set of Nash equilibria, strong Nash equilibria, and potential maximizers coincide:

Theorem 6.1 *On the class $\mathcal{C}$ of games, $SNE = NE = PM$.*

A proof of this result is given in parts. Recall that for any strategic game G, $SNE(G) \subseteq NE(G)$ and that for any exact potential game G, $PM(G) \subseteq NE(G)$. It therefore suffices to prove the following propositions.

Proposition 6.2 *For each game $G \in \mathcal{C}$, $NE(G) \subseteq PM(G)$.*

Proposition 6.3 *For each game $G \in \mathcal{C}$, $NE(G) \subseteq SNE(G)$.*

The proofs are based on the structure of the class $\mathcal{C}$ described in the previous section. We assume $n \in \mathbb{N}$ and a finite facility set F to be fixed. Each game $G \in \mathcal{C}(F, n)$ is given by a collection of vectors

$$((w_f(1), \ldots, w_f(n)))_{f \in F},$$

$$(w_f(1), \ldots, w_f(n)) \in \{v \in \mathbb{R}^n_+ \mid v_t \geqq v_{t+1} \text{ for all } t \in \{1, \ldots, n-1\}\}.$$

To compute the potential of Remark 4.7 it is necessary to add the utilities of the used facilities up to the number of users. This means that in each vector w_f all the first $n_f(x)$ numbers are added.

As a consequence it is clear that by n times consecutively choosing the facilities with highest remaining numbers, from left on, in the set of vectors $\{(w_f(1), \ldots, w_f(n))\}_{f \in F}$ a potential maximizing profile is found. This is illustrated in the following example.

Example 6.4 Let $G \in \mathcal{C}(\{f, g\}, 4)$ such that

$$w_f = (4, 3, 2, 1)$$
$$w_g = (5, 2, 1, 0)$$

In the first step we take the first cell in w_g, in the second step the first cell in w_f, in the third step the second cell of w_f and, finally, in the fourth step either the third cell of w_f or the second cell of w_g. Consequently, the potential maximizing strategy combinations are those $x \in F^N$ with $n_f(x) = 3$, $n_g(x) = 1$ and those with $n_f(x) = 2$, $n_g(x) = 2$. Notice that for these x, $P(x) = 14$ and that all Nash equilibria are potential maximizing. $\triangleleft$

Based on a switching argument the next lemma shows the similarities in utilities for different Nash equilibria.

Lemma 6.5 *Let $G \in \mathcal{C}(F, n)$ be determined by $((w_f(1), \ldots, w_f(n)))_{f \in F}$ and let x and y be Nash equilibria of G. For all $f, g \in F$ such that $n_f(x) < n_f(y)$ and $n_g(y) < n_g(x)$, and for all $l \in \{n_f(x) + 1, \ldots, n_f(y)\}$ and $m \in \{n_g(y) + 1, \ldots, n_g(x)\}$ it holds that*

$$w_f(l) = w_f(n_f(y)) = w_g(n_g(x)) = w_g(m).$$

Proof. Let $f, g \in F$ and l, m be as described in the lemma. Both x and y are Nash equilibria, so $w_f(n_f(y)) \geqq w_g(n_g(y)+1) \geqq w_g(m) \geqq w_g(n_g(x)) \geqq w_f(n_f(x)+1) \geqq w_f(l) \geqq w_f(n_f(y))$. □

This lemma is used in the proof of Proposition 6.2.

Proof. [**Proposition 6.2**] Let $G \in \mathcal{C}(F, n)$ be determined by $((w_f(1), \ldots, w_f(n)))_{f \in F}$. It suffices to show that $P(x) = P(y)$ if x is a Nash equilibrium and y a potential maximizing strategy combination. Let $x \in NE(G)$ and $y \in PM(G)$. Facilities $f \in F$ such that $n_f(x) = n_f(y)$ add as much to $P(x)$ as to $P(y)$. Furthermore, by Lemma 6.5, if $n_f(x) < n_f(y)$ and $n_g(y) < n_g(x)$ for certain $f, g \in F$ then $w_f(l) = w_f(n_f(y)) = w_g(n_g(x)) = w_g(m)$ for all $l \in \{n_f(x)+1, \ldots, n_f(y)\}$ and $m \in \{n_g(y)+1, \ldots, n_g(x)\}$. The total contribution of the facilities in the set $\{f \in F \mid \ n_f(x) \neq n_f(y)\}$ to the potentials $P(x)$ and $P(y)$ is apparently the same. □

Remains to prove Proposition 6.3.

Proof. [**Proposition 6.3**]. Let $G \in \mathcal{C}\ (F, n)$ be given by $((w_f(1), \ldots, w_f(n)))_{f \in F}$ and let $x \in NE(G)$. Suppose $S \subseteq N$ can strictly improve the payoff for all its members by switching to a strategy combination $y_S \in F^S$. Call the resulting strategy combination $y = (y_S, x_{N \setminus S})$. If $n_f(y) > n_f(x)$ for some $f \in F$, a player $i \in S$ exists such that $y_i = f$ and $x_i = g, g \neq f$. This implies $w_f(n_f(x)+1) \geqq w_f(n_f(y)) > w_g(n_g(x))$, which contradicts the fact that x is a Nash equilibrium. So $n_f(x) = n_f(y)$ for all $f \in F$. Therefore every player in S chooses a new facility already chosen by a member of S and obtains a higher utility. Among the utilities assigned to members of S there is a maximum, since S is finite. Any player in S rewarded with this maximum cannot get more in the new configuration. Hence a contradiction arises. Every Nash equilibrium is strong. □

In the last part of this section we consider strictly strong Nash equilibria. Recall that given a game $\langle N, (X_i)_{i \in N}, (u_i)_{i \in N} \rangle$, a strategy profile $x \in X$ is a *strictly strong Nash equilibrium* if for all coalitions $S \subseteq N$ and strategy combinations $y_S \in X_S$, $u_i(y_S, x_{N \setminus S}) = u_i(x)$ for all $i \in S$ or $u_i(y_S, x_{N \setminus S}) < u_i(x)$ for at least one $i \in S$. The following example illustrates that the properties of $\mathcal{C}$ do not guarantee the existence of strictly strong Nash equilibria.

Example 6.6 Consider the game $G \in \mathcal{C}(\{f, g\}, 3)$ with w_f, w_g given by

$$w_f = (4, \boxed{2}, 0),$$
$$w_g = (\boxed{3}, 2, 1),$$

where the squared numbers depict a strong Nash equilibrium payoff. If the two players choosing f agree that one of them switches to g and the other one sticks to f, the utility will still be 2 for the switching one but increases from 2 to 4 for the remaining player. A similar argument holds for the other type of strong Nash equilibria given by

$$w_f = (\boxed{4}, 2, 0)$$
$$w_g = (3, \boxed{2}, 1)$$

Since these are the only two types of strong Nash equilibria, and neither of them is strictly strong, strictly strong Nash equilibria do not exist. $\triangleleft$

7 Extensions of the model

The class $\mathcal{C}$ is characterized by properties (P1), (P4), and (P5). It is obvious that relaxation of those properties will have consequences on the result presented in Section 6.

First of all, the classes of congestion games of Quint and Shubik (1994), Milchtaich (1996), and Konishi *et al.* (1997) without (P5) not necessarily admit an exact potential.

Secondly, consider the class $\mathcal{CP}$ strategic games which satisfy the properties (P1) and (P5). Each n person game G in $\mathcal{CP}$ is a potential game and can be represented by a collection of arbitrary vectors $((w_f(1), \dots, w_f(n)))_{f \in F} \in (\mathbb{R}^n)^F$. It is obvious that not every game $G \in \mathcal{CP}$ has a strong Nash equilibrium. For instance, the Prisoner's Dilemma in Example 2.2 is an element of $\mathcal{CP}$ with $F = \{c, d\}$, $w_c = (4, 1)$ and $w_d = (0, 3)$, but does not have a strong Nash equilibrium. But even the existence of a strong Nash equilibrium for a game $G \in \mathcal{CP}$ does not guarantee that each Nash equilibrium is strong too, nor that a strong equilibrium is a potential maximizer. The next example gives a game $G \in \mathcal{CP}$ such that $\emptyset \neq SNE(G) \subset NE(G)$ and $SNE(G) \cap PM(G) = \emptyset$.

Example 7.1 Let $G \in \mathcal{CP}(\{f,g\},3)$ with

$$\begin{aligned} w_f &= (4, 0, \boxed{5}) \\ w_g &= (4, 2, 0) \end{aligned}$$

The unique strong Nash equilibrium in which all three players chooses facility f is indicated. By Proposition 4.6, the potential can be computed as in Remark 4.7. The maximal potential arises at the non strong equilibria which are given by

$$\begin{aligned} w_f &= (\boxed{4}, 0, 5) \\ w_g &= (4, \boxed{2}, 0) \end{aligned}$$

◁

Finally, consider the class of strategic games $\mathcal{C}'$ satisfying (P1'),(P4), and (P5). Similarly to Proposition 6.3 one can show

Theorem 7.2 *For every game $G \in \mathcal{C}'$, $NE(G) = SNE(G)$.*

This result coincides with that of Holzman and Law-Yone (1997, Theorem 2.1). In the class $\mathcal{C}'$, however, the set of potential maximizing strategy combinations need not coincide with the set of Nash equilibria, as can be seen in the following example.

Example 7.3 Consider the game $G \in \mathcal{C}'(\{f,g,h\},5)$ in which three players have strategy set $\{f,h\}$ and two $\{g,h\}$. The benefit vectors are

$$\begin{aligned} w_f &= (4, 2, \boxed{1}, -, -) \\ w_g &= (\boxed{3}, 2, -, -, -) \\ w_h &= (\boxed{2}, 1, 1, 0, 0) \end{aligned}$$

where the squared numbers depict a Nash equilibrium payoff. It represents strategy combinations in which the three players with strategy set $\{f,h\}$ all play f. Consider now the equilibrium in which two of those three play f and the other plays h.

$$\begin{aligned} w_f &= (4, \boxed{2}, 1, -, -) \\ w_g &= (3, \boxed{2}, -, -, -) \\ w_h &= (\boxed{2}, 1, 1, 0, 0) \end{aligned}$$

The potential can be computed as in Remark 4.7. For the first type of equilibrium in this example, the potential value equals $4 + 2 + 1 + 3 + 2 = 12$, which is less than $4+2+3+2+2 = 13$, the potential value associated to the second type of equilibrium. $\triangleleft$

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